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Density waves in traffic flow model with relative velocity

L. Yu[a] and Z.-K. Shi

College of Automation, Northwestern Polytechnical University, 710072 Xi' an, P.R. China

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Abstract. The car-following model of traffic flow is extended to take into account the relative velocity. The stability condition of this model is obtained by using linear stability theory. It is shown that the stability of uniform traffic flow is improved by considering the relative velocity. From nonlinear analysis, it is shown that three different density waves, that is, the triangular shock wave, soliton wave and kink-antikink wave, appear in the stable, metastable and unstable regions of traffic flow respectively. The three different density waves are described by the nonlinear wave equations: the Burgers equation, Korteweg-de Vries (KdV) equation and modified Korteweg-de Vries (mKdV) equation, respectively.

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1 Introduction

Traffic flow is an interesting phenomenon of modern world. Although we all experience it daily, traffic problems are far from being well understood. As typical signature of the complex behavior of traffic flow, traffic jams have been studied by various traffic models: car-following models, cellular automaton (CA) models, gas kinetic models and hydrodynamic models.

In recent years, many studies reveal traffic phenomena such as the phase transitions and the nonlinear waves. Kerner and Rehborn [1] observe the dynamical jamming transitions between the freely moving traffic at low density and jammed traffic at high density. The jamming transitions have the properties similar to the conventional phase transitions: the freely moving traffic and jammed traffic correspond to the gas and liquid phases, respectively.

In modern traffic theory, it is well-known that traffic jams occur in the high density region and propagate as various density waves. Some of these density waves show typical nonlinear waves such as the triangular shock wave, the soliton wave and the kink-antikink wave. The density waves are described by the nonlinear wave equations: the Burgers equation, the Korteweg-de Vries (KdV) equation and the modified Korteweg-de Vries (mKdV) equation, respectively. Recently, many researchers investigate the nonlinear wave equations for traffic flow by using the nonlinear analysis method. Kurtze and Hong [2] derive the KdV equation from the hydrodynamic model and show that the traffic soliton appears near the neutral stability line. Komatsu and Sasa [3] obtain the mKdV equation from the optimal velocity (OV) model proposed by Bando et al. [4] to describe the density waves in congestion. Muramatsu and Nagatani [5] derive the KdV equation from the OV model and describe traffic jams as soliton density waves.

For public demand, it is necessary to raise the transportation efficiency and prevent traffic jams. Many approaches to extending the traffic model toward suppressing the appearance of traffic congestion efficiently have been pursued. Jiang et al. [6] propose a full velocity difference model by taking into account the velocity difference. Xue [7] extends the OV model to take into account the relative velocity. Ge et al. [8] present an extended car-following model with consideration the headway of arbitrary number of vehicles ahead. Li et al. [9] present a forward looking relative velocity model by introducing relative velocities of arbitrary number of vehicles ahead into the full velocity difference model.

In this paper, an extended car-following model is proposed by considering the relative velocity. We consider the effect of relative velocity upon the phase transitions and the nonlinear waves by using the linear stability theory and nonlinear analysis. The linear stability analysis shows that the traffic flow of the extended model is more stable by taking into account the relative velocity. The Burgers equation, KdV equation and mKdV equation are derived to describe the density waves in the stable, metastable and unstable region of traffic flow by using nonlinear analysis, respectively. The triangular shock wave, soliton wave and kink-antikink wave appear as the density waves in the three regions, respectively.

The organization of this paper is as follows. In Section 2 we give the extended car-following model by considering the relative velocity. In Section 3 the stability

[a] e-mail: yuleijk@126.com

of the extended model is analyzed by the linear stability theory. The neutral stability lines and the critical points varying with the weighted coefficient of the relative velocity are shown. In Section 4 the Burgers equation, KdV equation and mKdV equation are derived to describe the density waves in the stable, metastable and unstable region of traffic flow by using nonlinear analysis. Section 5 give a summary.

2 Model

In general, the dynamic equation of the car-following model on the single-lane highway can be written as [10]

$$\ddot{x}_n = f_{\rm sti}(v_n, \Delta x_n, \Delta v_n), \tag{1}$$

where the function $f_{\rm sti}$ represents the response to the stimuli received by the nth vehicle. Equation (1) defines the acceleration or deceleration of the nth vehicle, which is determined by the stimuli received from the surrounding traffic. The stimuli are composed of the velocity v_n of the nth vehicle, the headway $\Delta x_n = x_{n+1} - x_n$ between two successive vehicles and the relative velocity $\Delta v_n = v_{n+1} - v_n$.

Newell [11] proposes a car-following model which is given by a first-order differential equation by introducing a delay time which plays an important role in the occurrence of traffic congestion. The motion of nth car is given as follows

$$\frac{dx_n(t+\tau)}{dt} = V(\Delta x_n(t)), \tag{2}$$

where $x_n(t)$ is the position of the nth vehicle at time t, $\Delta x_n(t)$ is the headway, $V(\Delta x_n(t))$ is the optimal velocity function and τ is the delay time.

Combining with the idea of equation (1), the optimal velocity function of equation (2) may be determined not only by the headway but also by the relative velocity. Then, an extended car-following model by considering the relative velocity is constructed as follows

$$\frac{dx_n(t+\tau)}{dt} = V(\Delta x_n(t), \Delta v_n(t)), \tag{3}$$

where $V(\Delta x_n(t), \Delta v_n(t))$ is the optimal velocity function formulated as a function of the headway and the relative velocity. The idea of the extended model (3) is that the driver adjusts the car velocity $dx_n(t+\tau)/dt$ according to the stimuli received by the nth vehicle, that is, the headway $\Delta x_n(t)$ and the relative velocity $\Delta v_n(t)$. The delay time τ is the time lag that it takes the car velocity to reach the optimal velocity $V(\Delta x_n(t), \Delta v_n(t))$ when the traffic flow is varying.

Let the optimal velocity function be a linear combination between the headway and the relative velocity, i.e., $V(\Delta x_n(t), \Delta v_n(t)) = V(\Delta x_n(t)) + \lambda \Delta v_n(t)$, where the weighted coefficient of the relative velocity λ is a constant independent of time, velocity and position ($0 \le \lambda \le 1$). Therefore, the extended model (3) can be rewritten as

$$\frac{dx_n(t+\tau)}{dt} = V(\Delta x_n(t)) + \lambda \Delta v_n(t). \tag{4}$$

If $\lambda = 0$, the extended model is the Newell model expressed by equation (2).

The optimal velocity function $V(\Delta x_n(t))$ is selected similar to that used by Bando et al. [4]

$$V(\Delta x_n(t)) = \frac{v_{\max}}{2}[\tanh(\Delta x_n(t) - h_c) + \tanh(h_c)], \tag{5}$$

where h_c is the safety distance and $v_{\max}$ is the maximal velocity. The optimal velocity function is a monotonically increasing function of the headway and has an upper bound (i.e. the maximal velocity). When the headway is less than the safety distance, the vehicle reduces its velocity to prevent from crashing into the preceding vehicle. On the other hand, if the headway is larger than the safety distance, the vehicle increases its velocity to the maximal velocity. The optimal velocity function given by equation (5) has the turning point (inflection point) at $\Delta x_n = h_c$, that is,

$$V''(h_c) = \left.\frac{d^2 V(\Delta x_n)}{d\Delta x_n^2}\right|_{\Delta x_n = h_c} = 0.$$

It is important that the optimal velocity function has the turning point. Otherwise, we cannot derive the mKdV equation which has the kink-antikink solution to describe the traffic jams.

The phase diagram in the parameter space $(\Delta x, 1/\tau)$ of equation (4) is shown in Figure 1. The solid line represents the coexisting curve which can be obtained from the solution of the mKdV equation (see Sect. 4.3). The neutral stability line (that is, the spinodal line) is shown by the dotted line. There exists the critical point $(h_c, 1/\tau_c)$ which is the apex of the neutral stability line similar to the conventional gas-liquid phase transition. In this figure, the traffic flow is divided into three regions: the first is the stable region above the coexisting curve, the second is the metastable region between the neutral stability line and the coexisting curve and the third is the unstable region below the neutral stability line. The traffic flow is stable above the neutral stability line and the traffic jams will not appear. While below the neutral stability line, traffic flow is unstable and the density waves emerge.

3 Linear stability analysis

We apply the linear stability theory to analyze the extended model expressed by equation (4). The stability of uniform traffic flow is considered, which can be defined by such a state that all cars move with the identical headway h and the optimal velocity $V(h)$. Then the solution $x_n^{(0)}(t)$ representing the uniform steady state of equation (4) can be written as

$$x_n^{(0)}(t) = hn + V(h)t. \tag{6}$$

Let $y_n(t)$ be a small deviation from the uniform steady state $x_n^{(0)}(t)$: $x_n(t) = x_n^{(0)}(t) + y_n(t)$. Substituting it into equation (4) and linearizing the resulting equation, we can obtain

$$\frac{dy_n(t+\tau)}{dt} = V'(h)\Delta y_n(t) + \lambda \frac{d\Delta y_n(t)}{dt}, \tag{7}$$

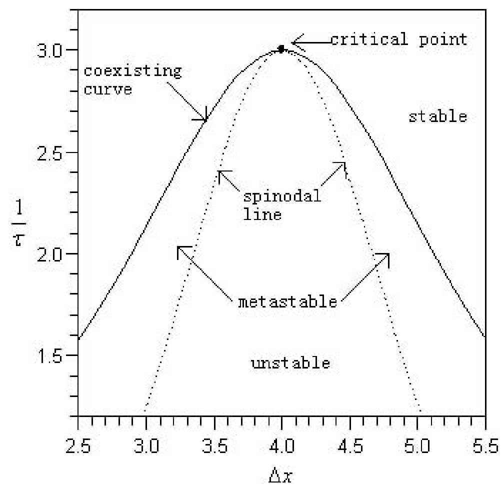

Fig. 1. The phase diagram of equation (4).

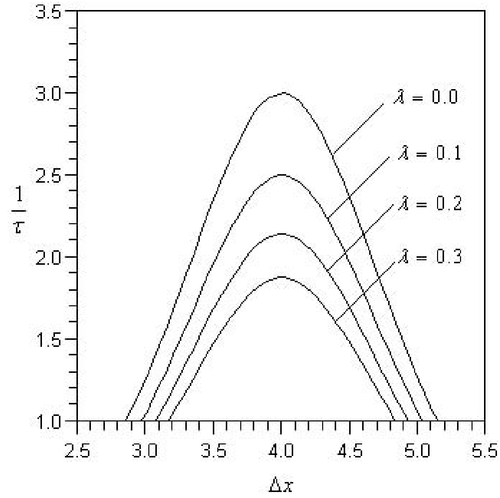

Fig. 2. Neutral stability lines vary with λ.

where $V'(h)$ is the derivative of the optimal velocity function $V(\Delta x_n(t))$ at $\Delta x_n(t) = h$ and $\Delta y_n(t) = y_{n+1}(t) - y_n(t)$. By expanding $y_n(t) \propto \exp(ikn + zt)$, the following equation of z is obtained

$$ze^{z\tau} - (V'(h) + \lambda z)(e^{ik} - 1) = 0. \quad (8)$$

By expanding $z = z_1(ik) + z_2(ik)^2 + ...$ and inserting it into equation (8), the first- and second-order terms of ik are obtained

$$z_1 = V'(h), \quad z_2 = \frac{V'(h)}{2} - \tau z_1^2 + \lambda z_1. \quad (9)$$

If z_2 is a negative value, the uniform steady state becomes unstable for long wavelength modes, while the uniform flow is stable when z_2 is a positive value. The neutral stability condition for the uniform steady state is given by

$$\tau_s = \frac{1 + 2\lambda}{2V'(h)}. \quad (10)$$

For small disturbances of long wavelength, the uniform traffic flow is stable if

$$\tau < \frac{1 + 2\lambda}{2V'(h)}. \quad (11)$$

Comparing the stability condition with that of the Newell model expressed by equation (2) [12], we conclude that equation (4) is stabilized in the region

$$\frac{1}{2\tau} < V'(h) < \frac{1 + 2\lambda}{2\tau}, \quad (12)$$

by the effect of the relative velocity, which means that by introducing the relative velocity into the car-following model, traffic flow becomes more stable similar to that by taking into account the headway of car ahead [8].

Figure 2 shows the neutral stability lines varying with the weighted coefficient of relative velocity in the parameter space $(\Delta x, 1/\tau)$ for equation (4). The solid lines represent the neutral stability lines for various values of λ. The apex of each curve indicates the critical point $(h_c, 1/\tau_c)$. From Figure 2, it can be seen that the critical points and the neutral stability lines are lowered with taking into account larger value of λ. So the extended model with taking into account the relative velocity expressed by equation (4) is more stable than the Newell model. Moreover, with increasing λ, that is, with attaching importance to the effect of the relative velocity, the stability of the uniform traffic flow is improved. The traffic jams are thus suppressed efficiently. When $\lambda = 0$, the results given in this section agree with those given in references [12,13].

4 Nonlinear analysis and density waves

Car density could fluctuate for various reasons to form density wave in traffic flow, which can lead to the traffic jams. Different nonlinear wave equations have been derived to describe the corresponding density waves from many models of traffic flow. Here we will investigate the nonlinear analysis to consider the slowly varying behaviors in the stable, metastable and unstable region of the extended model expressed by equation (4). For later convenience, equation (4) can be rewritten as

$$\frac{d\Delta x_n(t+\tau)}{dt} = V(\Delta x_{n+1}(t)) - V(\Delta x_n(t)) + \lambda\left[\frac{d\Delta x_{n+1}(t)}{dt} - \frac{d\Delta x_n(t)}{dt}\right]. \quad (13)$$

We now consider long-wavelength modes on coarse-grained scales. The simplest way to describe the long-wavelength modes is the long-wave expansion. We apply the reductive perturbation method to equation (13). We introduce slow scales for space variable n and time variable t and define slow variables X and T for $0 < \varepsilon \ll 1$

$$X = \varepsilon^p(n + bt), \quad T = \varepsilon^q t, \quad (14)$$

where b is a constant to be determined. We set the headway as

$$\Delta x_n(t) = h + \varepsilon^m R(X, T). \quad (15)$$

In the expressions of equations (14) and (15), p, q and m are parameters which will be determined for following discussion in the stable, metastable and unstable region.

Substituting equations (14) and (15) into equation (13) and making the Taylor expansions about ε, one obtains the following nonlinear partial differential equation

$$\begin{aligned}&\varepsilon^{p+m}b\partial_X R+\varepsilon^{q+m}\partial_T R+\varepsilon^{p+q+m}2b\tau\partial_X\partial_T R\\&+\varepsilon^{2p+m}b^2\tau\partial_X^2 R+\varepsilon^{3p+m}\frac{b^3\tau^2}{2}\partial_X^3 R+\varepsilon^{4p+m}\frac{b^4\tau^3}{6}\partial_X^4 R=\\&\varepsilon^{p+m}V'\partial_X R+\varepsilon^{2p+m}\frac{V'}{2}\partial_X^2 R+\varepsilon^{3p+m}\frac{V'}{3!}\partial_X^3 R\\&+\varepsilon^{4p+m}\frac{V'}{4!}\partial_X^4 R+\varepsilon^{p+2m}\frac{V''}{2}\partial_X R^2+\varepsilon^{2p+2m}\frac{V''}{4}\partial_X^2 R^2\\&+\varepsilon^{p+3m}\frac{V'''}{3!}\partial_X R^3+\varepsilon^{2p+3m}\frac{V'''}{12}\partial_X^2 R^3+\lambda\Big[\varepsilon^{2p+m}b\partial_X^2 R\\&+\varepsilon^{p+q+m}\partial_X\partial_T R+\varepsilon^{3p+m}\frac{b}{2}\partial_X^3 R+\varepsilon^{4p+m}\frac{b}{6}\partial_X^4 R\Big],\quad(16)\end{aligned}$$

where $V'=\frac{dV(\Delta x)}{d\Delta x}|_{\Delta x=h}, V''=\frac{d^2V(\Delta x)}{d\Delta x^2}|_{\Delta x=h}$ and $V'''=\frac{d^3V(\Delta x)}{d\Delta x^3}|_{\Delta x=h}$. For simplicity, V', V'' and V''' correspond to $V'(h), V''(h)$ and $V'''(h)$ in the above equation and hereafter.

4.1 Burgers equation in the stable flow region

In the stable traffic flow, the long-wave mode is considered by taking $p=1, q=2$ and $m=1$. Substituting the values of these parameters into equation (16), we obtain the following nonlinear partial differential equation

$$\begin{aligned}\varepsilon^2(b-V')\partial_X R+\varepsilon^3[\partial_T R-V''R\partial_X R\\+(b^2\tau-\frac{V'}{2}-\lambda b)\partial_X^2 R]=0.\quad(17)\end{aligned}$$

By taking $b=V'$, the second order term of ε is eliminated from equation (17) and we have

$$\partial_T R-V''R\partial_X R=\left(\frac{1+2\lambda}{2}-V'\tau\right)V'\partial_X^2 R.\quad(18)$$

In accordance with the stability condition (11) in Section 3, traffic flow is stable because the coefficient $((1+2\lambda)/2-V'\tau)$ of the second derivative in equation (18) is positive value. Thus, in the stable region, equation (18) is just the Burgers equation, of which the solution is a train of N-shock waves and is given by

$$\begin{aligned}R(X,T)=\frac{1}{|V''|T}\left[X-\frac{\eta_n+\eta_{n+1}}{2}\right]-\frac{\eta_{n+1}-\eta_n}{2|V''|T}\\\times\tanh\left[\left(\frac{1+2\lambda}{2}-V'\tau\right)\frac{(\eta_{n+1}-\eta_n)(X-\xi_n)}{4|V''|T}\right].\\(19)\end{aligned}$$

The coordinates of the shock fronts are given by ξ_n and η_n are the coordinates of the intersections of the slopes with the x-axis ($n=1,2,...,N$). The triangular shock waves propagate backward with the propagation velocity v_p which is given by

$$v_p=V'(h).\quad(20)$$

The shock waves propagate backwards only with respect to the moving vehicles, but the shock propagates forward in the absolute system, if one is in the region of the free traffic. The propagation velocity decreases with increasing the headway. According to equation (19), we can observe that $R(X,T)\to 0$ when $t\to\infty$, which means that any density wave in free traffic flow will evolve to a homogeneous flow in the course of time. When $\lambda=0$, the results given in this section agree with the results given in reference [12].

4.2 KdV equation in the metastable flow region

Next we consider the case of $p=1, q=3$ and $m=2$. In this section we derive the KdV equation near the neutral stability line which has the soliton solution. From equation (16), one obtains the following nonlinear partial differential equation

$$\begin{aligned}&\varepsilon^3(b-V')\partial_X R+\varepsilon^4\left(b^2\tau-\frac{V'}{2}-\lambda b\right)\partial_X^2 R\\&+\varepsilon^5\left[\partial_T R+\left(\frac{b^3\tau^2}{2}-\frac{V'}{6}-\frac{\lambda b}{2}\right)\partial_X^3 R-\frac{V''}{2}\partial_X R^2\right]\\&+\varepsilon^6\left[(2b\tau-\lambda)\partial_X\partial_T R+\left(\frac{b^4\tau^3}{6}-\frac{V'}{24}-\frac{\lambda b}{6}\right)\partial_X^4 R\right.\\&\left.-\frac{V''}{4}\partial_X^2 R^2\right]=0.\quad(21)\end{aligned}$$

Near the neutral stability line $\tau_s=(1+2\lambda)/(2V')$, $\tau=(1-\varepsilon^2)\tau_s$, and taking $b=V'$, the third- and fourth-order terms of ε are eliminated from equation (21). Then equation (21) can be rewritten as the simplified equation

$$\begin{aligned}&\varepsilon^5[\partial_T R-f_1\partial_X^3 R-f_2 R\partial_X R]\\&+\varepsilon^6[-f_3\partial_X^2 R+f_4\partial_X^4 R+f_5\partial_X^2 R^2]=0,\quad(22)\end{aligned}$$

where

$$\begin{aligned}f_1&=\frac{1-12\lambda^2}{24}V', f_2=V'', f_3=\frac{1+2\lambda}{2}V',\\f_4&=\frac{1-12\lambda^2-16\lambda^3}{48}V', f_5=\frac{1+2\lambda}{4}V''.\end{aligned}$$

In order to derive the standard KdV equation with higher order correction, we make the following transformations for equation (22)

$$T=\sqrt{f_1}T',\quad X=-\sqrt{f_1}X',\quad R=\frac{1}{f_2}R'.\quad(23)$$

By using of equation (23), we obtain the standard KdV equation with higher order correction term

$$\partial_{T'}R' + \partial^3_{X'}R' + R'\partial_{X'}R' + \frac{\varepsilon}{\sqrt{f_1}}\left[-f_3\partial^2_{X'}R' + \frac{f_4}{f_1}\partial^4_{X'}R' + \frac{f_5}{f_2}\partial^2_{X'}R'^2\right] = 0. \quad (24)$$

If we ignore the $O(\varepsilon)$ correction term in equation (24), it is just the KdV equation with the soliton solution

$$R'_0(X', T') = A\mathrm{sech}^2\left[\sqrt{\frac{A}{12}}\left(X' - \frac{A}{3}T'\right)\right]. \quad (25)$$

Amplitude A of the soliton solution is a free parameter. The perturbation term in equation (24) gives the condition of selecting a unique member from the continuous family of KdV solitons.

Assuming that $R'(X', T') = R'_0(X', T') + \varepsilon R'_1(X', T')$, we take into account the $O(\varepsilon)$ correction term. In order to determine the selected value of A for the soliton solution (25), it is necessary to satisfy the solvability condition

$$(R'_0, M[R'_0]) \equiv \int_{-\infty}^{+\infty} dX' R'_0 M[R'_0] = 0, \quad (26)$$

where $M[R'_0]$ is the $O(\varepsilon)$ term in equation (24). By performing the integration, we obtain the selected value of amplitude A for the soliton solution as follows

$$A = \frac{21 f_1 f_2 f_3}{24 f_1 f_5 - 5 f_2 f_4} = \frac{21(12\lambda^2 - 1)}{104\lambda^2 - 10\lambda - 7}V'. \quad (27)$$

By rewriting each variable to the original one, we obtain the soliton solution of the headway as the desired solution

$$\Delta x_n(t) = h + \frac{A}{f_2}\left|1 - \frac{\tau}{\tau_s}\right| \mathrm{sech}^2\left\{\sqrt{\frac{A}{12 f_1}\left|1 - \frac{\tau}{\tau_s}\right|} \times \left(n + V't + \frac{A}{3}\left|1 - \frac{\tau}{\tau_s}\right| t\right)\right\}. \quad (28)$$

Thus, the soliton density wave described by the KdV equation is obtained near the neutral stability line. When $\lambda = 0$, then $A = 3V'$. The results given in this section agree with the results given in reference [13].

4.3 mKdV equation in the unstable flow region

We derive the mKdV equation near the critical point, that is, the turning point (infection point) with $V''(h_c) = 0$. The critical point will help us to obtain the mKdV equation which has the kink-antikink solution to describe the traffic jams.

By taking $p = 1, q = 3$ and $m = 1$, we obtain the following nonlinear partial differential equation from equation (16)

$$\varepsilon^2(b - V')\partial_X R + \varepsilon^3\left(b^2\tau - \frac{V'}{2} - b\lambda\right)\partial^2_X R + \varepsilon^4\left[\partial_T R + \left(\frac{b^3\tau^2}{2} - \frac{V'}{6} - \frac{\lambda b}{2}\right)\partial^3_X R - \frac{V'''}{6}\partial_X R^3\right] + \varepsilon^5\left[(2b\tau - \lambda)\partial_X\partial_T R + \left(\frac{b^4\tau^3}{6} - \frac{V'}{24} - \frac{\lambda b}{6}\right)\partial^4_X R - \frac{V'''}{12}\partial^2_X R^3\right] = 0, \quad (29)$$

where $V' = \frac{dV(\Delta x)}{d\Delta x}|_{\Delta x = h_c}$ and $V''' = \frac{d^3V(\Delta x)}{d\Delta x^3}|_{\Delta x = h_c}$. Near the critical point $(h_c, 1/\lambda_c)$, $\tau = (1+\varepsilon^2)\tau_c$, and taking $b = V'$, the second- and third-order terms of ε are eliminated from equation (29). Then equation (29) can be rewritten as the simplified equation

$$\varepsilon^4[\partial_T R - g_1\partial^3_X R + g_2\partial_X R^3] + \varepsilon^5[g_3\partial^2_X R + g_4\partial^4_X R + g_5\partial^2_X R^3] = 0, \quad (30)$$

where

$$g_1 = \frac{1 - 12\lambda^2}{24}V', g_2 = -\frac{V'''}{6}, g_3 = \frac{1 + 2\lambda}{2}V',$$
$$g_4 = \frac{1 - 12\lambda^2 - 16\lambda^3}{48}V', g_5 = \frac{1 + 2\lambda}{12}V'''.$$

In order to derive the standard mKdV equation with higher order correction, we make the following transformations for equation (30)

$$T = \frac{1}{g_1}T', \quad R = \sqrt{\frac{g_1}{g_2}}R'. \quad (31)$$

Then we obtain the standard mKdV equation with higher order correction term

$$\partial_{T'}R' - \partial^3_X R' + \partial_X R'^3 + \frac{\varepsilon}{g_1}\left[g_3\partial^2_X R' + g_4\partial^4_X R' + \frac{g_1 g_5}{g_2}\partial^2_X R'^3\right] = 0. \quad (32)$$

If we ignore the $O(\varepsilon)$ correction term in equation (32), it is just the mKdV equation with the kink-antikink solution

$$R'_0(X, T') = \sqrt{B}\tanh\left[\sqrt{\frac{B}{2}}(X - BT')\right]. \quad (33)$$

Similar to the process of deriving the amplitude A for KdV equation in Section 4.2, we get the selected value of propagation velocity B for the kink-antikink solution as follows

$$B = \frac{5 g_2 g_3}{2 g_2 g_4 - 3 g_1 g_5} = \frac{120}{5 - 4\lambda - 52\lambda^2}. \quad (34)$$

By rewriting each variable to the original one, we obtain the kink-antikink solution of the mKdV equation

$$R(X, T) = \sqrt{\frac{g_1 B}{g_2}}\tanh\left[\sqrt{\frac{B}{2}}(X - B g_1 T)\right]. \quad (35)$$

Because we adopt the explicit form equation (5) for the optimal velocity function, $V' = v_{\max}/2$ and $V''' = -v_{\max}$. Then the amplitude C of the kink-antikink solution is given by

$$C = \sqrt{\frac{g_1 B}{g_2}\left|\frac{\tau}{\tau_c} - 1\right|}.$$

The kink-antikink density wave, solution of the mKdV equation, helps us obtain the coexisting curve in the phase diagram (see Fig. 1). The coexisting phase of traffic flow consists of the freely moving phase at low density and the jammed phase at high density. The headway of the freely moving phase and the jammed phase are given, respectively, by $\Delta x = h_c + C$ and $\Delta x = h_c - C$. Thus, we can obtain the coexisting curve in the $(\Delta x, 1/\tau)$ plane.

We also obtain the kink-antikink solution of the headway as the desired solution

$$\Delta x_n(t) = h_c + \sqrt{\frac{g_1 B}{g_2}\left|\frac{\tau}{\tau_c} - 1\right|}\tanh\left\{\sqrt{\frac{B}{2}\left|\frac{\tau}{\tau_c} - 1\right|} \times \left(n + V't - Bg_1\left|\frac{\tau}{\tau_c} - 1\right| t\right)\right\}. \quad (36)$$

Thus, the mKdV equation is derived near the critical point and its kink-antikink solution appears as density wave in the unstable region. When $\lambda = 0$, then $B = 24$. The results given in this section agree with the results given in reference [13].

From Section 4.1, we obtain that the density wave is described by the Burgers equation in the evolution of traffic flow from initial nonuniform to uniform distribution. With increasing density of traffic flow, the density wave, respectively, is described by the KdV equation and mKdV equation in the metastable region and unstable region from Sections 4.2 and 4.3. Near the critical point, the disturbance leads to traffic jams which is described by the kink-antikink density wave.

From above analysis for equation (4), the Burgers equation in the stable region, KdV equation in the metastable region and mKdV equation in the unstable region are derived respectively. It has been shown that only near the neutral stability line, does the soliton density wave appear. The soliton wave is the solution of the KdV equation. Only near the critical point, the kink-antikink density wave, which is the solution of the mKdV equation, is derived to describe the traffic jams.

5 Summary

We investigate an extended car-following model by considering the relative velocity. The stability condition and density waves of the extended model are obtained by using the linear stability theory and nonlinear analysis. It is shown that the stability of traffic flow is improved by considering the relative velocity. In addition, the density waves for the extended model are investigated by using the perturbation method in the stable, metastable and unstable regions of traffic flow. We obtain that nonuniform density profile evolves to the uniform density profile in the stable flow region by the triangular shock wave, which can be described by the Burgers equation. We also find that the density wave is respectively described by the KdV equation and mKdV equation in the metastable region and unstable region with increasing density of traffic flow. In this paper, it is shown that three different density waves, the triangular shock wave, soliton wave and kink-antikink wave, appear in the stable, metastable and unstable regions of traffic flow respectively. The three different density waves are described by the nonlinear wave equations: the Burgers equation, the KdV equation and the mKdV equation, respectively.

References

1. B.S. Kerner, H. Rehborn, Phys. Rev. Lett. **79**, 4030 (1997)
2. D.A. Kurtze, D.C. Hong, Phys. Rev. E **52**, 218 (1995)
3. T. Komatsu, S.I. Sasa, Phys. Rev. E **52**, 5574 (1995)
4. M. Bando, K. Hasebe, A. Nakayama, A. Shibata, Y. Sugiyama, Phys. Rev. E **51**, 1035 (1995)
5. M. Muramatsu, T. Nagatani, Phys. Rev. E **60**, 180 (1999)
6. R. Jiang, Q.S. Wu, Z.J. Zhu, Phys. Rev. E **64**, 017101 (2001)
7. Y. Xue, Chin. Phys. **11**, 1128 (2002)
8. H.X. Ge, S.Q. Dai, L.Y. Dong, Phys. A **365**, 543 (2006)
9. Z.P. Li, Y.C. Liu, Eur. Phys. J. B **53**, 367 (2006)
10. D. Chowdhury, L. Santen, A. Schreckenberg, Phys. Rep. **329**, 199 (2000)
11. G.F. Newell, Oper. Res. **9**, 209 (1961)
12. T. Nagatani, Phys. Rev. E **61**, 3564 (2000)
13. H.X. Ge, R.J. Cheng, S.Q. Dai, Phys. A **357**, 466 (2005)